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Multi-layer transient heat conduction using transition time scales

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Abstract

The solution of multi-layer transient heat conduction problems may be simplified by analyzing the different transition times of the various layers of a composite slab. These transition times, in fact, allow the 'disturbance' effect of each layer on the eigenvalues of the composite slab to be analyzed and estimated. It was found that the eigenvalues may be obtained in the first approximation by merging in increasing order the suitable corrected eigenvalues of each layer for which explicit equations are available. The errors in the resulting dimensionless temperatures are of one order of magnitude larger than the deviations between the exact and approximate eigenvalues. In particular, when one of the two outer boundaries is kept at constant temperature and the transition times of the layers are quite different, the eigenvalues may be written down very simply as the eigenvalues of the layer whose exposed surface is not at prescribed temperature. In this paper the temperature solution for the case of a two-layer slab with interlayer thermal contact resistance is presented.

Keywords: Composite slabs; Transient heat conduction; Eigenvalues; Transition time; Long-time solutions; Convergence criterion; Green's functions

1. Introduction

Two-layer composite slabs are of a great interest in many of today's engineering applications where transient heat conduction takes place. For example, a typical laser absorption calorimetry experiment [1] involves a composite region consisting of a thin glass sample (used in optical amplifiers connected to optical fibers) and an aluminum holder. Another area of interest is in the so-called double heat flux conductimeter [2] where the experimental apparatus is consisting of three parallel layers. However, having the two flux-meter plates of the same material and thickness and boundary conditions on two exposed surfaces of the same kind, the apparatus could be considered as a two-layer composite slab. Also, structures composed of one carbon matrix-carbon fiber material (orthotropic) and a layer of mica (isotropic) are in increasing usage in extreme temperature environments such as in aircraft and space operations [3,4].

Although some standard numerical methods are easily applied for the transient two-layer problems stated above, there are cases where the analytical solution [3–9] gives more reliable results and greater insight into the physical sense of the phenomena. Moreover, the emerging fields of verification and validation of numerically based approximate solutions [10] require to have accurate analytical solutions of transient two-layer heat conduction problems. As short-time solutions are available today for only single-layer bodies [10], the solutions for two-layer configurations are only in the form of long-time solutions and involve the presence of eigenvalues [3–8].

Therefore, in order that the two-layer analytical solution can in every respect be 'used', it is needed for practical purposes to compute the eigenvalues of the corresponding Sturm–Liouville problem. However, because of discontinuous coefficients due to piecewise-homogeneous bodies [11–15], this problem is not of traditional type. This discontinuity inevitably leads to irregularities within the entire spectrum of roots (eigenvalues) of the eigenvalue equation. Contrary, the are no similar irregularities when homogeneous bodies are considered. In such cases the Sturm–Liouville coefficients are perfectly smooth which makes possible the computing of eigenvalues by approximate equations [11,16–20]. As a matter of fact, explicit solutions were derived also for systems with discontinuous coefficients. They were given in the form of an asymptotic scheme and accurate only for large eigenvalues. However, as those solutions were de-

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Nomenclature

a_i	thickness of the ith layer ..... m
Bi_c	Biot number at the interface $x = 0 : 1/R_c$
Bi_i	Biot number for the ith layer: $h_i a_1/k_1$
$(c\rho)_i$	heat capacity per unit volume for the ith layer ..... $\mathrm{J\,K^{-1}\,m^{-3}}$
C_e	accuracy constant
e_i	thermal effusivity of the ith layer: $\sqrt{k_i(c\rho)_i}$ ..... $\mathrm{W\,s^{0.5}\,m^{-2}\,K^{-1}}$
$f(\beta)$	function on the LHS of the two-layer eigencondition (6)
$\Im_{\zeta,m}$	mth corrective factor (for z_m) defined by Eq. (16)
h_i	heat transfer coefficient for the ith layer ..... $\mathrm{W\,m^{-2}\,K^{-1}}$
h_c	thermal contact conductance ..... $\mathrm{W\,m^{-2}\,K^{-1}}$
k_i	thermal conductivity of the ith layer . $\mathrm{W\,m^{-1}\,K^{-1}}$
R_c	dimensionless thermal contact resistance: $k_1/(h_c a_1)$
t	time ..... s
t^+	dimensionless time: t/t_{c1}
t_{ci}	transition time scale for the ith layer: a_i^2/α_i .... s
T_i	temperature of the ith layer ..... K
T_0	uniform initial temperature ..... K
x	space coordinate ..... m
$X_{i,m}$	mth eigenfunction of the ith layer
z_{1,m_1}	m_1th dimensionless eigenvalue of the 1st layer, i.e. m_1th root of Eq. (12)
z_{2,m_2}	m_2th dimensionless eigenvalue of the 2nd layer, i.e. m_2th root of Eq. (14)
z_m	mth initial approximation for β_m

Greek symbol

α_i	thermal diffusivity of the ith layer ..... $\mathrm{m^2\,s^{-1}}$
β_m	mth dimensionless eigenvalue of the two-layered slab of Fig. 1: $\lambda_m \cdot a_1$
γ	geometric ratio: a_2/a_1
δ	thermal diffusivity ratio: α_2/α_1
ε	thermal effusivity ratio: $e_2/e_1 = \kappa/\sqrt{\delta}$
ζ_m	mth approximation for β_m
Θ_i	dimensionless temperature of the ith layer: T_i/T_0
κ	thermal conductivity ratio: k_2/k_1
λ_m	mth eigenvalue of the two-layered slab of Fig. 1 ..... $\mathrm{m^{-1}}$
ν_{1,m_1}	location of the m_1th asymptote of $f(\beta)$ due to the 1st layer
ν_{2,m_2}	location of the m_2th asymptote of $f(\beta)$ due to the 2nd layer
ξ	dimensionless space coordinate: x/a_1
τ_c	transition time scale ratio: $t_{c2}/t_{c1} = \gamma^2/\delta$

Subscripts

i	index (i.e., 1 or 2)
1	first layer and left side ($x = -a_1$) of the two-domain slab of Fig. 1
2	second layer and right side ($x = a_2$) of the two-layer slab of Fig. 1

veloped for general mathematical studies [11] and for a variety of non-heat-conduction applications, such as applied mechanics [12], geophysics [14] and oceanics [15], they are not suitable for computing the eigenvalues in the case of transient heat conduction problems in layered media. For example, the equations given in Refs. [11,15] are applicable only when thermal contact resistance is negligible and the values of thermal diffusivity are same for both layers.

The objectives of the present work are:

(1) to compute the eigenvalues of a two-layer slab with finite thermal contact resistance using explicit approximate equations,
(2) to provide a criterion for estimating the maximum number of eigenvalues, and
(3) to analyze the errors in final results.

For these objectives, a semi-analytical procedure has been developed. It is based on the evaluation of different transition time period of each layer. Namely, it was found that the eigenvalues of both homogeneous regions (for which explicit approximate equations are available in literature [11,16–20]) keep almost completely (through their transition times) the main physical information contained in the composite slab for computing its eigenvalues ('*physical part*' of the procedure). Then, the use of a corrective factor in the explicit form completes the above physical information and allows in a single step that the initial approximation be very close to the exact value of the eigenvalue ('*analytical part*' of the procedure). Similar results may be obtained for the resulting temperatures over broad ranges of those dimensionless groupings (namely, Bi_1, Bi_2, ε, $\sqrt{\tau_c}$ and R_c) on which the eigenvalues depend. Also, it was found that the different transition time scales of two layers affect the maximum number of eigenvalues and, hence, the convergence criterion of exponentially-converging temperature solutions.

2. Temperature solution in a two-layer slab

The temperature solution of a linear one-dimensional time-dependent heat conduction problem in the case of two-layered slab with finite thermal contact resistance (Fig. 1), initially at temperature $F(x)$ ($F_1(x)$ for the 1st and $F_2(x)$ for the 2nd layer) and subject with homogeneous boundary conditions of any kind, can be evaluated for $t^+ \geqslant 0$ as [5]

$$T_i\left(\xi, t^+\right) = A_i \sum_{m=1}^{\infty} c_m X_{i,m}(\xi) \exp\left(-\beta_m^2 t^+\right) \quad (i = 1, 2) \qquad (1)$$

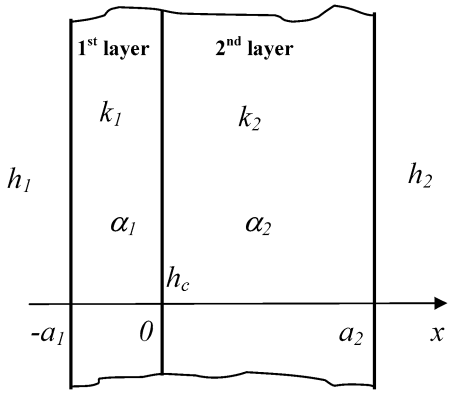

Fig. 1. Schematic representation of a 1-D two-region slab (with thermal contact resistance) subject to convective boundary conditions.

where $A_1 = 1$ and $A_2 = 1/\varepsilon$. Coefficients c_m depend on the initial temperature as follows

$$c_m = \frac{1}{N_m}\left[\int_{\xi'=-1}^{0} F_1(\xi')X_{1,m}(\xi')\,\mathrm{d}\xi' + \frac{1}{\sqrt{\delta}}\int_{\xi'=0}^{\gamma} F_2(\xi')X_{2,m}(\xi')\,\mathrm{d}\xi'\right] \tag{2}$$

$$N_m = \frac{1}{2}\left\{\left[1+\Pi_1^2(\beta_m)\right]\left(1+\frac{Bi_1}{\beta_m^2+Bi_1^2}\right) + \frac{1+\Pi_2^2(\beta_m)}{\varepsilon}\left[\sqrt{\tau_c}+\frac{Bi_2/\varepsilon}{\beta_m^2+(Bi_2/\varepsilon)^2}\right]+R_c\right\} \tag{3}$$

By setting $Bi_i \to \infty$ or $Bi_i = 0$ in the last equations, we have boundary conditions of the 1st or 2nd kind, respectively. Notice that the exponential part of Eq. (1) depend on a dimensionless time t^+ defined as t/t_{c1} for both the layers, where t_{c1} is the 'transition time' [21] of the 1st region. Also, the temperature of the second region ($i = 2$) depends explicitly on the thermal effusivity ratio ε. The latter is due to the application of Tittle's approach [22] and separation of variables of the heat diffusion equation, as was done in the early paper [5] (where, however, the perfect thermal contact between two layers was assumed). If the problem involves heat generation ($g_1(x,t)$ for the 1st region and $g_2(x,t)$ for the 2nd region) as well as nonhomogeneous boundary conditions ($f_1(t)$ at $x = -a_1$ and $f_2(t)$ at $x = a_2$), the temperature solution may be calculated using Green's functions [23], that is,

$$\begin{aligned} T_i(\xi,t^+) &= \int_{\xi'=-1}^{0} G_{i1}\big(\xi,\xi',u^+\big)\big|_{u^+=t^+}F_1(\xi')\,\mathrm{d}\xi' + \int_{\xi'=0}^{\gamma} G_{i2}\big(\xi,\xi',u^+\big)\big|_{u^+=t^+}F_2(\xi')\,\mathrm{d}\xi' \\ &\quad + \left(\frac{a_1^2}{k_1}\right)\int_{u^+=0}^{t^+}\left[\int_{\xi'=-1}^{0} G_{i1}\big(\xi,\xi',u^+\big)g_1\big(\xi',t^+-u^+\big)\,\mathrm{d}\xi' + \frac{\delta}{\kappa}\int_{\xi'=0}^{\gamma} G_{i2}\big(\xi,\xi',u^+\big)g_2\big(\xi',t^+-u^+\big)\,\mathrm{d}\xi'\right]\mathrm{d}u^+ \\ &\quad + \left(\frac{a_1}{k_1}\right)\left[\int_{u^+=0}^{t^+} G_{i1}\big(\xi,\xi'=-1,u^+\big)f_1\big(t^+-u^+\big)\,\mathrm{d}u^+ + \frac{\delta}{\kappa}\int_{u^+=0}^{t^+} G_{i2}\big(\xi,\xi'=\gamma,u^+\big)f_2\big(t^+-u^+\big)\,\mathrm{d}u^+\right] \end{aligned} \tag{4}$$

where $u^+ = t^+ - \tau^+$ is the so-called cotime. Appropriate replacements have to be made in the last term of Eq. (4) when the boundary conditions are of the 1st kind [23]. Notice that when the two-layer problems involve local heat generations, the temperatures may also be obtained by using the Fourier transform [9]. The Green's functions $G_{ij}(\xi,\xi',u^+)$ which appear in Eq. (4) are obtainable from the solution of the homogeneous version of the problem under consideration (i.e. solution (1)). Therefore, comparing Eq. (1) and the first term of Eq. (4) one obtains

$$G_{ij}\big(\xi,\xi',u^+\big) = B_{ij}\sum_{m=1}^{\infty}\frac{1}{N_m}X_{i,m}(\xi)X_{j,m}(\xi')\exp\big(-\beta_m^2u^+\big) \quad (i,j=1,2) \tag{5}$$

where $B_{11} = \kappa$, $B_{12} = \varepsilon$, $B_{21} = \kappa/\varepsilon$ and $B_{22} = 1$. The eigenvalue problem associated with the present two-layer transient problem may concisely be denoted by X3(C3)3, where 3 indicates the order of boundary condition, while (C3) indicates an imperfect contact interface (see more detail in [23] for the notation system proposed by Beck et al.). In particular, the quantity $\beta = \lambda \cdot a_1$ appearing in Eq. (1) through Eq. (5) is any root (dimensionless eigenvalue) other than zero of the following transcendental equation

$$f(\beta) = \Pi_1(\beta) + \frac{1}{\varepsilon}\Pi_2(\beta) + \beta R_c = 0 \tag{6}$$

where the functions $\Pi_1(\beta)$ and $\Pi_2(\beta)$ may be taken as

$$\Pi_1(\beta) = \frac{\beta + Bi_1\tan(\beta)}{Bi_1 - \beta\tan(\beta)} \tag{7a}$$

$$\Pi_2(\beta) = \frac{\beta + (Bi_2/\varepsilon)\tan(\sqrt{\tau_c}\beta)}{(Bi_2/\varepsilon) - \beta\tan(\sqrt{\tau_c}\beta)} \tag{7b}$$

The contact conductance h_c appearing in $R_c = 1/Bi_c$ may be calculated as proposed in [24]. The roots of the eigencondition (6), instead, are all real. In particular, the negative roots are equal in absolute value to the positive ones and there are no repeated eigenvalues as $\partial f/\partial\beta > 0$. They form also a monotonically increasing infinite series according to Sturm–Liouville systems with piecewise functions [11–15]. Thus, there are numerous space-variable eigenfunctions $X_{1,m}(\xi)$ and $X_{2,m}(\xi)$, each corresponding to a consecutive value of β_m ($m = 0, 1, 2, \ldots$)

$$X_{1,m}(\xi) = \sin(\beta_m \xi) + \Pi_1(\beta_m)\cos(\beta_m \xi) \quad \xi \in [-1, 0] \tag{8a}$$

$$X_{2,m}(\xi) = \sin(\beta_m \xi/\sqrt{\delta}) - \Pi_2(\beta_m)\cos(\beta_m \xi/\sqrt{\delta}) \quad \xi \in [0, \gamma] \tag{8b}$$

They vanish for $\beta_{m=0} = 0$ ('trivial' eigenvalue: notice that the first term of the 1D summation of Eq. (5) is characterized by the subscript $m = 1$) and satisfy the orthogonality property given in Ref. [5]. From transcendental equation (6) it follows that the eigenvalues β_m depend on five dimensionless variables, namely Bi_1, Bi_2, ε, $\sqrt{\tau_c}$ and R_c. Therefore, eigenvalues and Green's functions are completely independent of:

(1) origin adopted for the Cartesian frame of reference;
(2) initial temperature distribution;
(3) nonhomogeneity of the heat diffusion equation (due to heat sources or sinks), and
(4) nonhomogeneity of the boundary conditions.

The computation of the eigenvalues is the fundamental step in order that the formal solutions (1) and (4) can be 'used' to calculate the thermal field within each layer.

3. Computation of eigenvalues

The $f(\beta)$ function has infinite vertical asymptotes, whose locations ν may be determined as solutions to the following two equations

$$\beta \tan(\beta) - Bi_1 = 0 \tag{9a}$$

$$(\sqrt{\tau_c}\beta)\tan(\sqrt{\tau_c}\beta) - (Bi_2\sqrt{\tau_c}/\varepsilon) = 0 \tag{9b}$$

Following Haji-Sheikh and Beck's approach [6], Eq. (9a) is the same eigencondition for a single-layer slab with thickness a_1 and thermal conductivity k_1, subject to linear boundary conditions of 3rd and 2nd kind at $x = -a_1$ (characterized by h_1) and $x = 0$, respectively. That case is concisely denoted by X32. Eq. (9b), instead, is the same transcendental equation for a single-layer plate with thickness $a_2/\sqrt{\delta}$ and thermal conductivity k_2, subject to the boundary condition of the 2nd kind at $x = 0$ and the linear boundary condition of Robin type at $x = a_2/\sqrt{\delta}$ (characterized by $h_2\sqrt{\delta}$). That case is denoted by X23 but the eigenvalue transcendental equation is formally the same [23].

The roots of Eqs. (9) and, hence, the locations ν_m of the vertical asymptotes of $f(\beta)$, may readily be computed by using the explicit approximate relations proposed in [16–20]. The computation through Burniston and Siewert's equations [16], subsequently re-proposed by Leathers and McCormick [17], involves, however, the numerical integration. As the $f(\beta)$ function increases monotonically between its asymptotes (whose locations are independent of R_c), it may be stated that: (1) the $f(\beta)$ function intersects the β-axis infinite times, and (2) there exists only one eigenvalue between any two adjacent asymptotes. Therefore, the asymptotes $\nu_{L,m}$ and $\nu_{U,m}$ of $f(\beta)$ may be assumed as lower and upper bounds for the eigenvalues β_m of Eq. (6), as was done by Haji-Sheikh and Beck [6]. Once the bounds are known, the eigenvalues may be reached through any root-finding iteration. Concerning this, to avoid that $f(\nu_{L,m})$ and $f(\nu_{U,m}) \to \infty$ when the above iteration is applied, the eigencondition (6) has to be rewritten in a free-asymptote format, that is, containing only sines and cosines.

However, analyzing the 'disturbance' effect of each layer on the other one through the transition time ratio τ_c, it can be noted that the explicit approximate equations for computing the eigenvalues of homogeneous bodies [11,16–20] may properly be extended to two-layered slabs, as shown in next sections. The availability of explicit relations, in fact, simplifies significantly the treatment, reduces considerably the computational time, provides a greater insight to the physical sense of the eigenvalues and, finally, gives improved confidence in numerical results.

3.1. Analysis of the transition time scale ratio

The development of a general procedure based on explicit approximate equations for the eigenvalues β_m requires the analysis of the role played by the dimensionless parameter τ_c on the reciprocal 'disturbance' effect of two layers.

The square root of this parameter appears in the function $\Pi_2(\beta)$ defined through Eq. (7b). As that function, which characterizes the eigenfunction (8b) related to the 2nd layer, is present in the transcendental equation (6), it is evident that $\sqrt{\tau_c}$ affects the contribution of the 2nd layer to the locations of the asymptotes of $f(\beta)$ function. Notice that the LHS of Eq. (9b) is the denominator of $\Pi_2(\beta)$. Contrary, the contribution of the 1st layer to the locations of the asymptotes of $f(\beta)$ function is not affected by $\sqrt{\tau_c}$. In such a case, in fact, the LHS of Eq. (9a) is the denominator of the $\Pi_1(\beta)$ function appearing in Eq. (6), where $\Pi_1(\beta)$ characterizes the eigenfunction (8a) related to the 1st layer.

As first and second boundary conditions include the extremes of the convective boundary conditions, the $f(\beta)$ asymptote locations ν_{1,m_1} and ν_{2,m_2} due to the 1st and 2nd layers are, respectively, in the following ranges

$$\nu_{1,m_1} \in \left[(m_1 - 1)\pi, (m_1 - 1/2)\pi\right] \quad m_1 = 1, 2, 3, \ldots \tag{10a}$$

$$\nu_{2,m_2} \in \left[\frac{(m_2 - 1)\pi}{\sqrt{\tau_c}}, \frac{(m_2 - 1/2)\pi}{\sqrt{\tau_c}}\right] \quad m_2 = 1, 2, 3, \ldots \tag{10b}$$

The lower limits of the intervals (10) have been obtained setting $Bi_1 = 0$ and $Bi_2 = 0$ in Eqs. (9a) and (9b), respectively (boundary condition of the 2nd kind). The upper limits of the same intervals, instead, have been derived setting $Bi_1 \to \infty$ and $Bi_2 \to \infty$ in the same equations (boundary condition of the 1st kind).

It follows from Eqs. (10) that, when $\sqrt{\tau_c} < 1$, the spacing between any two adjacent asymptotes of the $f(\beta)$ function is mainly due to the first layer. In other words, until the asymptote locations due to the 1st layer are not 'disturbed' by those due to the 2nd layer, the above spacing is only due to the first region, and for this reason holds quite regular over the β-axis. Now, bearing in mind that there exists only one eigenvalue between any two neighboring asymptotes, it may be stated that the

Table 1
First twenty intervals (lower and upper bounds—first two columns) where the first twenty exact eigenvalues β_m of the eigencondition (6) are located for $Bi_1 = 1$, $Bi_2 = 2$, $R_c = 3$, $\sqrt{\tau_c} = 0.2$ and $\varepsilon = 50$. Approximations ζ_m (ζ_{1,m_1} and ζ_{2,m_2}) for β_m in a single step (i.e. after one iteration)

m	$\nu_{L,m}$	$\nu_{U,m}$	β_m	ζ_{1,m_1}—Eqs. (12) and (16)			ζ_{2,m_2}—Eqs. (14) and (16)		
1	0.44661818	0.86033359	0.4658409945759	–	–	–	0.4675149550865	1	UB
2	0.86033359	3.42561846	1.0665052270397	1.0663491793683	1	LB	–	–	–
3	3.42561846	6.43729818	3.5136653415080	3.5136649112273	2	LB	–	–	–
4	6.43729818	9.52933441	6.4874757842725	6.4874757800089	3	LB	–	–	–
5	9.52933441	12.64528722	9.5637914835748	9.5637914815317	4	UB	–	–	–
6	12.64528722	15.72068533	12.6714053633543	12.6714053486765	5	UB	–	–	–
7	15.72068533	15.77128487	15.7221671990973	–	–	–	15.7223513643256	2	UB
8	15.77128487	18.90240996	15.7929359183753	15.7929176592979	6	LB	–	–	–
9	18.90240996	22.03649673	18.9199855158436	18.9199855119272	7	LB	–	–	–
10	22.03649673	25.17244633	22.0515821031239	22.0515821029891	8	LB	–	–	–
11	25.17244633	28.30964285	25.1856586190756	25.1856586189717	9	UB	–	–	–
12	28.30964285	31.42229144	28.3213934628154	28.3213934615407	10	UB	–	–	–
13	31.42229144	31.44771464	31.4230334787159	–	–	–	31.4231256835866	3	UB
14	31.44771464	34.58642422	31.4586174125270	31.4586082053453	11	LB	–	–	–
15	34.58642422	37.72561283	34.5960534078279	34.5960534071697	12	LB	–	–	–
16	37.72561283	40.86517033	37.7344405609678	37.7344405609400	13	LB	–	–	–
17	40.86517033	44.00501792	40.8733201121571	40.8733201121334	14	UB	–	–	–
18	44.00501792	47.12813355	44.0125858659987	44.0125858656628	15	UB	–	–	–
19	47.12813355	47.14509774	47.1286283812241	–	–	–	47.1286898662644	4	UB
20	47.14509774	50.28536634	47.1523759318319	47.1523697839593	16	LB	–	–	–

LB = Lower bound for the dimensionless eigenvalue of Eq. (6).
UB = Upper bound for the dimensionless eigenvalue of Eq. (6).

eigenvalues β_m of the two-layered slab are regularly spaced until the asymptote locations due to the 1st layer are 'undisturbed' by the presence of those due to the 2nd layer. That explains the pattern of the eigenvalue spacing shown in the 3rd column of Table 1. In particular, two adjacent eigenvalues $\beta_{1+6\cdot p}$ and $\beta_{2+6\cdot p}$ (marked in gray) are very close between them, and this occurs cyclically (i.e. for $p = 0, 1, 2, \ldots$). Similar considerations (although in a diametrically opposite direction) may be drawn when $\sqrt{\tau_c} > 1$. In such a case, in fact, the spacing between any two adjacent asymptotes is mainly due to the second layer. When $\sqrt{\tau_c} = 1$, instead, two layers contribute to the locations of the $f(\beta)$ asymptotes in the same manner and, hence, 'disturb' reciprocally and continuously over the entire β-axis. Then the eigenvalue spacing is fully irregular, contrary to what has been maintained so far for Sturm–Liouville problems with discontinuous coefficients [11–15].

It is evident, therefore, that the effect of each layer on the spacing of both asymptotes and eigenvalues of the two-region slab depends strictly on the transition time scale ratio.

3.2. Explicit approximate equations for the eigenvalues

The physical insights deriving from the previous subsection are of notably concern because for example, when $\sqrt{\tau_c} < 1$, the eigenvalues z_{1,m_1} ($m_1 = 1, 2, 3, \ldots$) of the 1st layer can also be assumed as initial approximations for the eigenvalues β_{1,m_1} of the two-layered slab until the asymptote locations ν_{1,m_1} due to the 1st layer are not 'disturbed' by ν_{2,m_2} due to the 2nd one. This would also indicate that the effect of the second layer (through its inherent variables Bi_2/ε and $\sqrt{\tau_c}$) on the calculation of the initial approximations for β_{1,m_1} is nearly negligible. When the disturbance effect arrives, the eigenvalues z_{2,m_2} ($m_2 = 1, 2, 3, \ldots$) of the second layer can be assumed as initial approximations for the eigenvalues β_{2,m_2} of the two-layered slab. In such a case, the effect of the first layer (through its inherent variable Bi_1) on the calculation of the initial approximations for β_{2,m_2} can be neglected. Contrary, when $\sqrt{\tau_c} > 1$. Instead, when $\sqrt{\tau_c} = 1$, the eigenvalues z_{1,m_1} and z_{2,m_2} of the 1st and 2nd layers, respectively, can alternatively be assumed as initial approximations for the eigenvalues β_{1,m_1} and β_{2,m_2} of the two-layered slab.

The considerations above are the key to extend the explicit approximate equations for the eigenvalues of single-region slabs [11,16–20] to two-region slabs, whose eigenvalues may thus be obtained in the first approximation merging both z_{1,m_1} and z_{2,m_2}. For this purpose, the eigenvalues z_{1,m_1} of the 1st layer are the roots of the transcendental equation

$$\Pi_1(z_1) + z_1 R_c = 0 \tag{11}$$

obtained by setting $\Pi_2(\beta) = 0$ in Eq. (6), where $\Pi_2(\beta)$ characterizes the 2nd layer. The computation of z_{1,m_1} may be obtained by considering that Eq. (11) is the eigencondition of a single-layer plate with thickness a_1, thermal conductivity k_1, and linear boundary conditions of the 3rd kind at both $x = -a_1$ (with h_1) and $x = 0$ (with h_c)—X33 case. In fact, Eq. (11) may also be rewritten as

$$\tan(z_1) = \frac{z_1(Bi_1 + Bi_c)}{z_1^2 - Bi_1 Bi_c} \tag{12}$$

whose roots z_{1,m_1} ($m_1 = 1, 2, 3, \ldots$) may readily and accurately be obtained by using Haji-Sheikh and Beck's explicit approximate equations [19]. Hochstadt's solutions [11] may also be used but solely for large eigenvalues in view of their asymptotic structure. Burniston and Siewert's equations [16] (subsequently re-proposed by Leathers and McCormick [17]), Stevens and Luck's relations [18] as well as Marinetti and Vavilov's equations [20] cannot be exploited in such a case because they refer

to only one adiabatic (or with prescribed temperature) and one convective boundary condition, i.e. X23 and X13 (or X32 and X31) cases.

Similarly, the eigenvalues z_{2,m_2} of the 2nd layer are the roots of the following equation

$$\Pi_2(z_2) + z_2 R_c \varepsilon = 0 \tag{13}$$

As the function $\Pi_1(\beta)$ affects the 1st layer, the above equation has simply been obtained by setting $\Pi_1(\beta) = 0$ in Eq. (6). The computation of z_{2,m_2} may be derived using the fact that Eq. (13) is the eigencondition of a single-layer slab with thickness $a_2/\sqrt{\delta}$, thermal conductivity k_2, and linear boundary conditions of Robin type at both $x = 0$ (characterized by $h_c\sqrt{\delta}$) and $x = a_2/\sqrt{\delta}$ (characterized by $h_2\sqrt{\delta}$) (X33 case). After some algebraic steps, Eq. (13) becomes

$$\tan(\sqrt{\tau_c} z_2) = \frac{(\sqrt{\tau_c} z_2)(Bi_c\sqrt{\tau_c}/\varepsilon + Bi_2\sqrt{\tau_c}/\varepsilon)}{(\sqrt{\tau_c} z_2)^2 - (Bi_c\sqrt{\tau_c}/\varepsilon)(Bi_2\sqrt{\tau_c}/\varepsilon)} \tag{14}$$

whose roots z_{2,m_2} ($m_2 = 1, 2, 3, \ldots$) may be derived through explicit approximate relations [19].

3.3. Corrective factor

The Sturm–Liouville eigenvalue system of two layers appropriate for the temperature solutions (1) and (4) is given by

$$\frac{\mathrm{d}^2 X_1}{\mathrm{d}x^2} + \lambda^2 \cdot X_1 = 0 \tag{15a}$$

$$\frac{\mathrm{d}^2 X_2}{\mathrm{d}x^2} + \frac{\lambda^2}{\delta} \cdot X_2 = 0 \tag{15b}$$

$$\mp k_i \cdot \left(\frac{\mathrm{d}X_i}{\mathrm{d}x}\right)_{x=\mp a_i} + h_i \cdot X_i(x = \mp a_i) = 0 \tag{15c}$$

$$X_1(x = 0) + \frac{k_1}{h_c} \cdot \left(\frac{\mathrm{d}X_1}{\mathrm{d}x}\right)_{x=0} = X_2(x = 0) \tag{15d}$$

$$k_1 \cdot \left(\frac{\mathrm{d}X_1}{\mathrm{d}x}\right)_{x=0} = k_2 \cdot \left(\frac{\mathrm{d}X_2}{\mathrm{d}x}\right)_{x=0} \tag{15e}$$

where the negative sign in Eq. (15c) is valid for $i = 1$, while the positive sign for $i = 2$. By integrating the eigenvalue differential equations (15a) and (15b) on $\xi \in [-1, 0]$ and $\xi \in [0, \gamma]$, respectively, and summing up the resulting expressions consistently with the inner and outer boundary conditions (15c)–(15e), an appropriate corrective factor $\Im_{\zeta,m} = \Im_\zeta(z_m)$ of z_m may be derived in following explicit form

$$\Im_{\zeta,m} = \left[1 - \frac{f(z_m)}{z_m N(z_m)}\right]^{1/2} \tag{16}$$

Thus, the initial approximation z_m (i.e. z_{1,m_1} and z_{2,m_2}) for β_m (i.e. β_{1,m_1} and β_{2,m_2}) can be brought very close to the exact value β_m in a single step (i.e. after one iteration). In other words, the single-step approximation ζ_m for β_m may be obtained in a completely explicit form as $\zeta_m = z_m \Im_{\zeta,m}$, where z_m may be considered as the '*physical part*' of the solution while $\Im_{\zeta,m}$ as its '*analytical part*'.

If $f(z_m) > 0$, that is, z_m is an upper bound for β_m, then from Eq. (16) it follows that $\Im_{\zeta,m} < 1$ and so $\zeta_m < z_m$ and becomes closer to β_m. Contrary, when $f(z_m) < 0$. It is also interesting to observe that the corrective factor (16) depends on five dimensionless variables, namely Bi_1, Bi_2, ε, $\sqrt{\tau_c}$ and R_c, which are the *sole* variables affecting the β_m eigenvalues.

3.4. Special cases

It can be useful to note that there are two special cases when the eigenvalue computation may be performed efficiently and satisfactorily:

(1) the contact resistance is negligible ($R_c = 0$) and two layers of the composite slab have different thermal properties (i.e. $\kappa \neq 1$ and $\delta \neq 1$) but the same thermal effusivities ($\varepsilon = 1$);
(2) the thermal effusivity ratio is such that the relation $Bi_1 = Bi_2/\varepsilon$ is satisfied and the two layers of the composite slab are characterized by the same transition times ($\tau_c = 1$). Notice that the relation $Bi_1 = Bi_2/\varepsilon$ is satisfied independently of the value given to ε when either $Bi_1, Bi_2 \to \infty$ (X1(C3)1 case) or $Bi_1 = Bi_2 = 0$ (X2(C3)2 case).

In the first case, applying the compound angle formula for tangents and after some algebraic steps, the eigenvalue equation (6) reduces to

$$\tan\left[\beta(1 + \sqrt{\tau_c})\right] = \frac{[\beta(1 + \sqrt{\tau_c})][Bi_1(1 + \sqrt{\tau_c}) + Bi_2(1 + \sqrt{\tau_c})]}{[\beta(1 + \sqrt{\tau_c})]^2 - [Bi_1(1 + \sqrt{\tau_c})][Bi_2(1 + \sqrt{\tau_c})]} \tag{17}$$

whose roots β_m may be known using the explicit approximate equations given in Ref. [19]—X33 case. Then, for a two-domain slab with $\varepsilon = 1$ and $R_c = 0$, it may stated that the eigenvalues are regularly spaced (which is typical of single-layer slabs in view of the material homogeneity, in the sense of material 'regularity'), independently of the values given to $\sqrt{\tau_c}$, Bi_1 and Bi_2. This claim is important from a conceptual standpoint. In fact, it is in disagreement with what has commonly been asserted so far for Sturm–Liouville eigenvalue systems with discontinuous coefficients [11–15]: "*the more the coefficients suffer large discontinuities, the more severe these irregularities are*". In the current case, in fact, also when two layers are characterized by very different transition time scales, the roots continue to be regularly spaced.

In the second case, instead, after some appropriate manipulations and knowing that $R_c = 1/Bi_c$, the eigenvalue equation (6) becomes

$$\tan(\beta) = \frac{\beta[Bi_1 + Bi_c(1 + 1/\varepsilon)]}{\beta^2 - Bi_1[Bi_c(1 + 1/\varepsilon)]} \tag{18}$$

whose roots β_m may be derived using the explicit approximate equations given in [19]—X33 case. Then, for a two-region slab with $\tau_c = 1$ and $Bi_1 = Bi_2/\varepsilon$, considerations similar to those already drawn for the previous case are valid. In the current case, however, even when the interlayer resistance R_c is very large and the two layers are characterized by very different thermal effusivities, the roots continue to be regularly spaced.

4. Number of eigenvalues

When the Green's functions from Eq. (5) are evaluated, the convergence is algebraic and slow for zero time, but exponential and fast for dimensionless times other than zero. Because for nonhomogeneous boundary conditions and volumetric energy terms as well the Green's functions have always to be evaluated at zero time (see the time integration in Eq. (4)), the solution has very slow convergence (many terms are needed for an accurate computation).

The problem of slow convergence of the Green's functions may simply be by-passed by reformulating the heat conduction problem as the difference between 'steady state' and transient components (alternative GF solution method—AGFSM), as described in Section 3.4 of Ref. [23]. In other words, nonhomogeneous boundary conditions and heat generation terms may be removed and the problem becomes simplified with only one modified initial condition. In such a way, corresponding integral does not have to be evaluated at zero time, so the convergence has an exponential rate.

From what has been said, the search for the maximum number M of terms (convergence criterion) in summation (1) is a subject of great concern and will be treated in the next subsection.

4.1. Convergence criterion for exponentially-converging temperature solutions

For this purpose, the long-time Green's functions may be rewritten as

$$G_{ij}\left(\xi,\xi',t^+\right) = B_{ij}\sum_{m=1}^{M}\frac{1}{N_m}X_{i,m}(\xi)X_{j,m}(\xi')\exp\left(-\beta_m^2 t^+\right) + E_{ij}\left(\xi,t^+,M\right)\quad (i,j=1,2) \tag{19}$$

where M denotes the maximum number of terms (eigenvalues) in the above summation as well as $M+1$ indicates the maximum number of asymptotes of the $f(\beta)$ function. The exponential term in Eq. (19) reduces to the order of 10^{-10} ($\Rightarrow$ accuracy with absolute errors E_{ij} less than 10^{-10}) if the argument C_e of the exponential becomes as large as 23. That is, $\exp(-C_e) \cong 10^{-10}$ when $C_e = 23$ [10]. Therefore, M can be found in accordance with the accuracy given for the temperature and consistently with the dimensionless time of interest. In fact, the larger the time, the fewer terms are needed for a given accuracy. Contrary, the smaller the time, the larger number of terms is needed for the same accuracy.

To get the above convergence criterion, that proposed by McMasters et al. [10] for exponentially converging temperature solutions in homogeneous bodies may be extended to two-layered solids. However, as two domains have in general different transition times, Eq. (19) needs to be rewritten as

$$\begin{aligned} G_{ij}\left(\xi,\xi',t^+\right) &= B_{ij}\sum_{m_1=1}^{M_1}\frac{1}{N_{m_1}}X_{i,m_1}(\xi)X_{j,m_1}(\xi')\exp\left(-\beta_{m_1}^2 t^+\right) \\ &\quad + B_{ij}\sum_{m_2=1}^{M_2}\frac{1}{N_{m_2}}X_{i,m_2}(\xi)X_{j,m_2}(\xi')\exp\left(-\beta_{m_2}^2 t^+\right) \\ &\quad + E_{ij}\left(\xi,t^+,M\right)\quad (i,j=1,2) \end{aligned} \tag{20}$$

where M_1 and M_2 are the maximum numbers of terms due to the 1st and 2nd layers, respectively, with $M_1 + M_2 = M$. Now, as the eigenvalues z_{1,m_1} and z_{2,m_2} of the 1st and 2nd homogeneous layers can be assumed as initial approximations for the eigenvalues β_m (i.e. β_{1,m_1} and β_{2,m_2}) of the two-layered slab and the eigenvalues of homogeneous layers may be assumed approximately proportional to $(m-1)\cdot\pi$ [10], the following conclusions may be drawn:

- the β_{1,m_1} eigenvalues may be considered approximately proportional to $(m_1-1)\cdot\pi$ for a large index m_1. Then the maximum exponent in the 1-D m_1 summation of Eq. (20) becomes $[(M_1-1)\cdot\pi]^2\cdot t^+ = C_e$, which can be solved for M_1 to get

$$M_1 = \frac{1}{\pi}\left(\frac{C_e}{t^+}\right)^{1/2} + 1 = \frac{1}{\pi}\left(\frac{t_{c1}\cdot C_e}{t}\right)^{1/2} + 1 \tag{21a}$$

- the β_{2,m_2} eigenvalues may be considered approximately proportional to $(m_2-1)\cdot\pi/\sqrt{\tau_c}$ as m_2 increases. The maximum exponent in the 1-D m_2 sum of Eq. (20) may be taken then as $[(M_2-1)\cdot\pi/\sqrt{\tau_c}\,]^2\cdot t^+ = C_e$, which can be solved for M_2 to obtain

$$M_2 = \sqrt{\tau_c}(M_1-1) + 1 = \frac{1}{\pi}\left(\frac{t_{c2}\cdot C_e}{t}\right)^{1/2} + 1 \tag{21b}$$

Eqs. (21) allow the contribution of either region to the maximum number of asymptotes of $f(\beta)$ to be also known. It is given by M_1+1 and M_2+1, respectively. As the maximum number of asymptotes is $M+1$, one of the last two asymptotes (the greatest between ν_{1,M_1+1} and ν_{2,M_2+1}) has to be removed. Combining Eqs. (10) and (21) it is easy to prove that ν_{2,M_2+1} is larger than ν_{1,M_1+1} when $\tau_c < 4/9$ (or when $\tau_c < 1$ and $Bi_2 \to \infty$). Also, when the exposed boundary of the 2nd layer is kept at constant temperature ($Bi_2 \to \infty \Rightarrow$ X3(C3)1 case) and the square root of the transition time scale ratio verifies the following relation

$$\sqrt{\tau_c} < \frac{1}{2M_1+1} \tag{22}$$

the first asymptote location $\nu_{2,1}$ is greater than ν_{1,M_1+1}. This indicates that the second layer does not give any contribution to the maximum number of asymptotes of the $f(\beta)$ function. Furthermore, as $\beta_{2,1} > \nu_{2,1}$ (Section 3), the second layer does not give any contribution even to the maximum number of eigenvalues of the two-layered slab. In such a case, these eigenvalues may be written very simply as the eigenvalues of the first region governed by the transcendental equation (12) and may practically be considered undisturbed by the presence of the second material. Similarly, it may be proven that the ν_{1,M_1+1} location is greater than ν_{2,M_2+1} when $\tau_c > 9/4$ (or when $\tau_c > 1$ and $Bi_1 \to \infty$). Also, when the outer sidewall of the 1st layer has

a prescribed temperature ($Bi_1 \to \infty \Rightarrow$ X1(C3)3 case) and the transition time scale ratio satisfies the following constraint

$$\sqrt{\tau_c} > 2M_2 + 1 \tag{23}$$

the eigenvalues of the two-layered slab may be written very simply as the eigenvalues of the second region which has Eq. (14) as an eigencondition. In such a case, they may be considered undisturbed by the presence of the first material.

5. Numerical results and discussion

To show that the fully explicit solutions ζ_{1,m_1} and ζ_{2,m_2} defined in Sections 3.2 and 3.3 can be assumed as quite good approximations for the exact β_m eigenvalues (β_{1,m_1} and β_{2,m_2}), consider a two-region unsteady problem characterized by $Bi_1 = 1$, $Bi_2 = 2$, $R_c = 3$, $\sqrt{\tau_c} = 0.2$ and $\varepsilon = 50$, whose results are provided in Table 1 for $t^+ = 0.01$ and $C_e = 23$. The first two columns provide us intervals where the first twenty exact eigenvalues β_m of the two-region body (third column) are located. The lower $\nu_{L,m}$ and upper $\nu_{U,m}$ bounds of these intervals are the asymptotes of the $f(\beta)$ function, as illustrated in Section 3. In particular, the bounds marked in gray are the asymptotes ν_{2,m_2} of the $\Pi_2(\beta)$ function on the LHS of Eq. (13). This indicates that the 2nd layer contributes to the locations of the asymptotes of $f(\beta)$ only through four locations and in perfect accordance with the low value given to $\sqrt{\tau_c}$ (Section 4.1). The remaining bounds, instead, are the asymptotes ν_{1,m_1} of the $\Pi_1(\beta)$ function on the LHS of the eigencondition (11). This indicates that the 1st layer contributes through a good seventeen locations.

The fourth and fifth columns of Table 1, from the other side, give the approximate eigenvalues ζ_{1,m_1} and ζ_{2,m_2} derived from the 1st and 2nd layers of the composite slab, respectively. This indicates that the 1st domain contributes to the first twenty β_m through its first sixteen corrected eigenvalues ζ_{1,m_1} ($m_1 = 1, 2, \ldots, 16$). The 2nd domain, instead, contributes only through its first four corrected eigenvalues ζ_{2,m_2} ($m_2 = 1, \ldots, 4$). Notice that these results are in perfect accordance with the physical significance of transition time [21]. In fact, if $\sqrt{\tau_c} = 0.2$, the 2nd layer is much closer to a lumped capacitance body than the 1st one. This indicates that its thermal behavior requires less eigenvalues than those needed for the analysis of the 1st layer. It is also useful to observe that ζ_{1,m_1} and ζ_{2,m_2} are located only in those ranges whose lower limits are the asymptotes that correspond to the 1st and 2nd layers of the composite body, respectively.

A comparison between the approximate ζ_m and true β_m eigenvalues shows that the ζ_{1,m_1} can be assumed as quite good approximations for β_{1,m_1} until the asymptote locations ν_{1,m_1} are not 'disturbed' by those ν_{2,m_2}. In such a case, the eigenvalues may be derived with an error less than $10^{-7}\%$, as shown in Fig. 2(a). When the disturbance effect arrives, the ζ_{2,m_2} can be assumed as quite good approximations for β_{2,m_2}. Fig. 2(a) shows the percent deviation $|(\zeta_m - \beta_m)/\beta_m|$ for different m (from $m = 1$ to $m = 20$) and three different values of parameter R_c. The oscillating trend of the percent deviations is due to the disturbance effect of each layer on the other one. Notice that the errors for $R_c = 5$ are much lower than those for $R_c = 1$. This indicates that the disturbance effect of the 2nd layer on the calculation of β_{1,m_1} becomes more negligible as R_c increases. Thus, the thermal contact resistance R_c plays its well-known role of 'screen' not only for the heat diffusion between two layers, but also for the mutual influence of two layers.

On the other hand, Fig. 2(b) gives a convergence plot for $R_c = 3$ showing how the relative error $|(\zeta_m - \beta_m)/\beta_m|$ in the most slowly converging eigenvalues ($m = 1, 7, 13, 19$, as shown in Fig. 2(a)) decreases as the number of iterations increases. The above eigenvalues are all due to the second region and are completely disturbed by the presence of the first region. It has been found that the percent deviation reduces rapidly with the number of iterations N and is less than the order of magnitude of 10^{-10} for $N = 5$ for the above four eigenvalues. This indicates that the rate of convergence due to the corrective factor $\Im_{\zeta,m}$ defined by Eq. (16) is quite good, in particular for $N \leqslant 5$.

5.1. Errors in the resulting dimensionless temperatures

Now, it is interesting to determine the effect that the approximate eigenvalues computed above have on the temperature within each layer of the composite slab for $R_c = 3$. For this purpose, the homogeneous version of solution (4), i.e. the exponentially-converging temperature solution (1), has been considered. In such a case, assuming a uniform initial temperature distribution ($F_1(x) = T_0$ and $F_2(x) = T_0$), constants c_m defined by Eq. (2) become [5]

$$c_m = \frac{T_0}{N_m \beta_m}\big[\cos(\beta_m) - \cos(\sqrt{\tau_c}\beta_m) + \Pi_1(\beta_m)\sin(\beta_m) - \Pi_2(\beta_m)\sin(\sqrt{\tau_c}\beta_m)\big] \tag{24}$$

As $1/\sqrt{\delta} = \sqrt{\tau_c}/\gamma$ in Eq. (8b), the dimensionless temperature of the 2nd layer depends also on the geometric parameter γ (for example, we can consider $\gamma = 2$). Fig. 2(c) shows the percent difference in $\Theta_i = T_i/T_0 \in [0, 1]$ between using exact β_m and approximate roots ζ_m after one iteration and for three different dimensionless times t^+. The error in temperature is less than 6% (for first layer and $t^+ = 1$). As the first eigenvalue (the most important) is evaluated with an error of about 0.36%, it may be said, for this case, that the percent error in temperature is increased of one order of magnitude. The percent deviations between β_m and ζ_m, instead, affect the calculation of the temperature of the 2nd layer with an error having the same order of magnitude than the error characterizing the first eigenvalue. This is due to the fact that the first eigenvalue derives from the 2nd layer (see Table 1) and, hence, it is much more 'appropriate' to describe the thermal field in the 2nd region rather than that in the 1st one. Following the convergence procedure developed in Section 4.1 and assuming an accuracy of 10^{-15}, the above dimensionless temperatures have been calculated from the first twenty five terms of Eq. (1) for $t^+ = 0.01$, the early nine terms for $t^+ = 0.1$ and first four for $t^+ = 1$. Notice that step changes in the percent deviations at the interface $\xi = 0$ are due to the contact resistance. Finally, Fig. 2(d) shows how

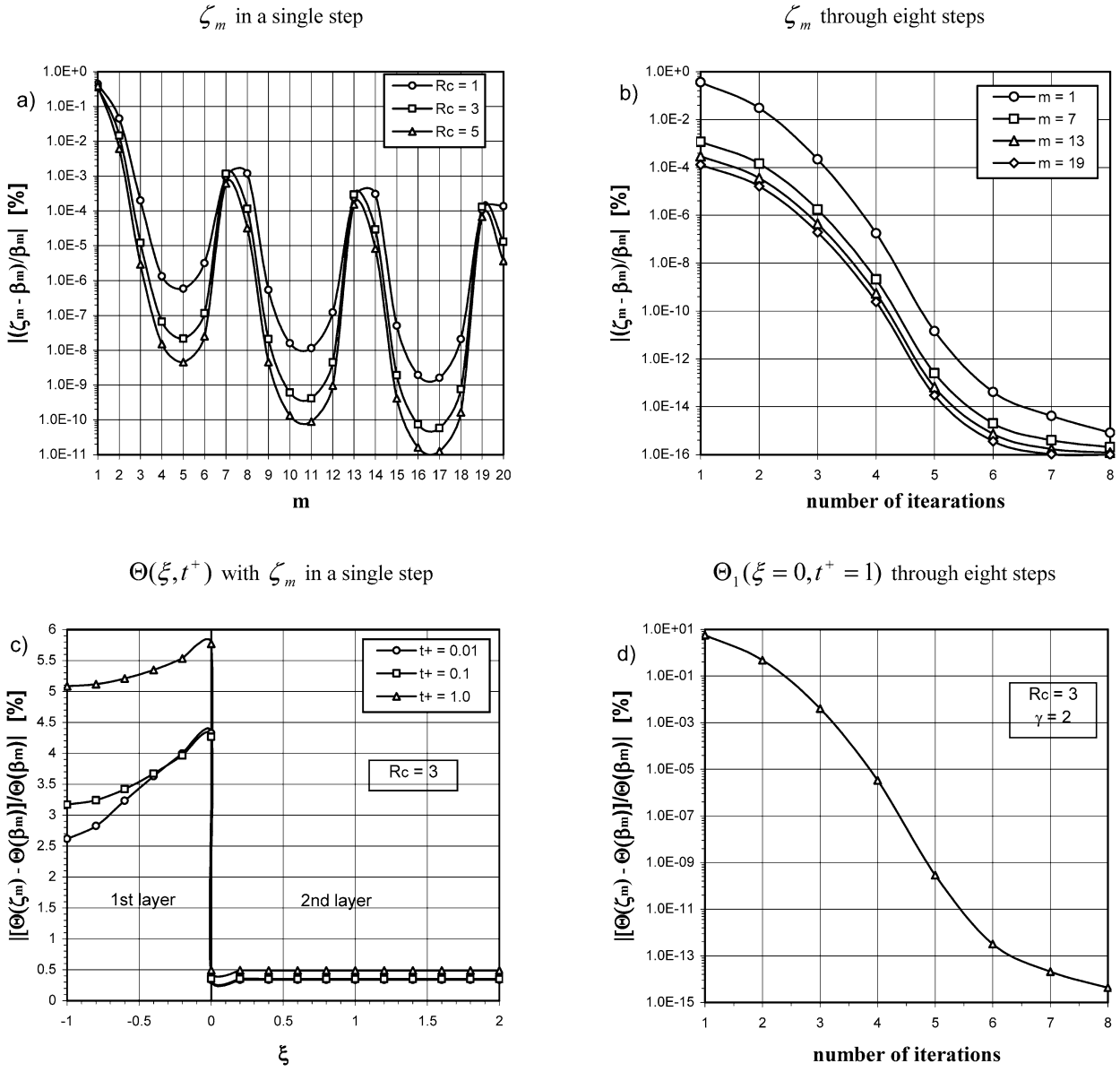

Fig. 2. Eigenvalue and temperature percent deviations when $Bi_1 = 1$, $Bi_2 = 2$, $\sqrt{\tau_c} = 0.2$ and $\varepsilon = 50$. (a) ζ_m in a single step for $m = 1$ through 20 with R_c as a parameter; (b) convergence plot of four eigenvalues for $R_c = 3$; (c) temperature after one iteration for $\xi \in [-1, \gamma = 2]$ when $R_c = 3$ with t^+ as a parameter; (d) convergence plot of the 1st layer temperature at the interface $\xi = 0$ when $R_c = 3$, $\gamma = 2$ and $t^+ = 1$.

the percent global error in the temperature predictions, namely $|[\Theta(\zeta_m) - \Theta(\beta_m)]/\Theta(\beta_m)|$, behaves with increasing the precision of the eigenvalues. This plot was dealing with the first-layer temperature calculated at the interface $\xi = 0$ for $t^+ = 1$, which represents the most slowly converging configuration (see Fig. 2(c)). It may be observed that the percent deviation reduces rapidly with the number of iterations N and is less than 10^{-9} as $N = 5$.

The procedure proposed above has been successfully tested for $m = 1$ through 100 and over broad ranges of the dimensionless groupings affecting eigenvalues and temperatures: Bi_1 and $Bi_2 \in [10^{-4}, 10^4]$, ε and $\sqrt{\tau_c} \in [10^{-2}, 10^2]$, $R_c \in [0, 10]$ and $\gamma \in [10^{-2}, 10^2]$. Moreover, the limiting boundary conditions of the first and second kind were successfully tested.

5.2. Boundary surface at fixed temperature

From Eqs. (21), it is easy to prove that when $t^+ = 0.01$, $\sqrt{\tau_c} = 0.03$, and $Bi_2 \to \infty$ the second layer does not give any contribution to the maximum number of asymptotes and eigenvalues of a two-layered slab for 10^{-10} accuracy (i.e. $C_e = 23$). The first layer, instead, provides 16 eigenvalues and 17 asymptotes. Fig. 3(a) shows that the percent deviations between the exact β_m and approximate z_m eigenvalues of the two-layered slab are very low, where z_m are the exact eigenvalues of the first layer (z_{1,m_1}). Furthermore, the above percent deviations do not have any oscillating trend for $m = 1, 2, \ldots, 16$, contrary to what was occurring in Fig. 2(a). All this is due to the fact that the eigenvalues β_m are not disturbed by the presence

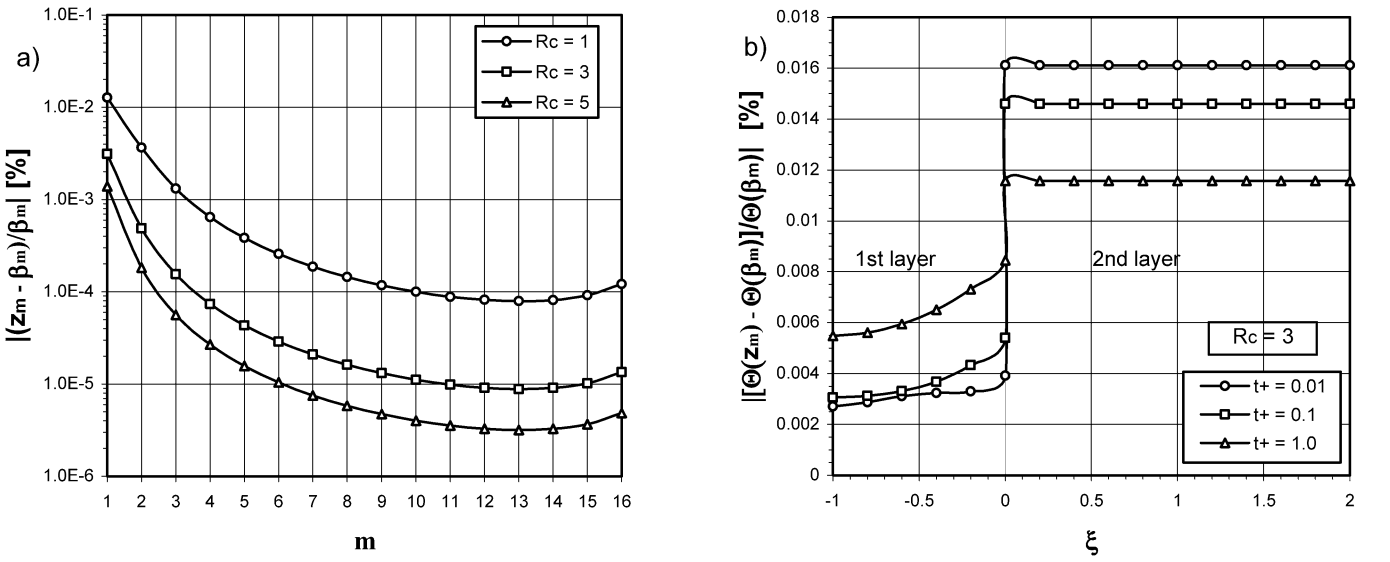

Fig. 3. Eigenvalue and temperature percent deviations when $Bi_1 = 1$, $Bi_2 \to \infty$, $\sqrt{\tau_c} = 0.03$ and $\varepsilon = 50$ (with $t^+ = 0.01$ and $C_e = 23$). (a) The $|(z_m - \beta_m)/\beta_m|$ percent deviation as a function of the index m when $R_c = 1, 3, 5$. (b) The $|[\Theta(z_m) - \Theta(\beta_m)]/\Theta(\beta_m)|$ percent deviation as a function of ξ with t^+ as a parameter when $R_c = 3$ and $\gamma = 2$.

of the 2nd layer. Notice, additionally, that the three deviations for $R_c = 1, 3, 5$ are minimized for $m = 13$. This indicates that, for $m > 13$, the initial approximations z_{1,m_1} for β_{1,m_1} begin to 'feel' the effect of the 2nd layer, that is, the disturbance effect caused by the first asymptote location due to the 2nd layer ($\nu_{2,m_2=1} = 52.35$) is more significant. Concerning this, it should be noted that $z_{1,m_1=16}$ (equal to 47.1663 for $R_c = 1$ and 47.1493 for $R_c = 5$) is very close to $\nu_{2,m_2=1}$. Finally, because β_m are not disturbed by the presence of the 2nd layer, the eigenvalues are regularly spaced within the *finite* numerable spectrum $m = 1, 2, \ldots, 16$. However, they continue to be irregularly spaced within the *entire* numerable spectrum due to the inevitable disturbance effect of the 2nd layer.

Fig. 3(b) shows that the consequent error in temperature (with $\gamma = 2$) is less than 0.017% (for second layer and $t^+ = 0.01$) and the percent deviations between β_m and z_m affect mainly the calculation of the temperature of the 2nd layer. This is due to the fact that, in the current case, all the eigenvalues derive from the 1st layer and hence are much more 'appropriate' to describe the thermal field in the 1st material rather than that in the 2nd one. Finally, as the percent errors in temperature are very low, the correction of the initial approximation z_m through the explicit factor (16) may be avoided. Thus, the eigenvalues of the two-layer slab may be written very simply as the eigenvalues of a homogeneous layer.

6. Concluding remarks

A semi-analytical procedure has been developed to extend the explicit approximate relations available in literature for the regularly spaced eigenvalues of a classical Sturm–Liouville problem to the irregularly spaced eigenvalues of an eigenproblem with coefficients having the step change at an interior point. A similar boundary-value eigenproblem arises from the transient heat conduction in a two-layered composite material.

The reciprocal 'disturbance' effect of two layers using their transition time scales has been investigated. The analysis showed that the eigenvalues of both regions can hold nearly completely the physical information contained in the piecewise-homogeneous slab. So, the two-layer eigenvalues may be obtained in first approximation by merging in increasing order the eigenvalues of each layer. Then, the use of a corrective factor in the explicit form completes the above physical information and allows the initial approximation to be brought very close to the exact value of the eigenvalue in just five steps (percent deviation less than 10^{-10}%).

The errors in the resulting dimensionless temperatures are, in general, one order of magnitude larger than the deviations between the exact and approximate eigenvalues. In particular, it was found that the temperature profile of the homogeneous layer of a two-layer material is very sensitive to eigenvalue errors when the eigenvalues are derived from the other layer.

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